Random Walks in Slightly Changing Environments

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Introduction

Main Object: Random Walks in Changing Environments (RWCEs): At vertex x, the RWCE jumps to a neighboring vertex y with probabilities proportional to edge-weights. Moreover, the edge-weights are **random variables** changing after each timestep.

Definition 1 (RWCE)**.** A *Random Walk in Changing Environments* on a graph $G = (V, E)$ is a stochastic process $\{\langle X_t, w_t \rangle\}_{t=0}^{\infty}$ such that for any $y \in V$, we have

$$\mathbb{P}(X_{t+1} = y \mid \mathcal{F}_t) = \frac{w_t(X_t, y)}{w_t(X_t)}$$

where $X_t \in V$, $w_t \in \mathcal{W}_E$, and $\mathcal{F}_t = \sigma(X_0, \ldots, X_t, w_0, \ldots, w_t)$ for each $t \geq 0$.

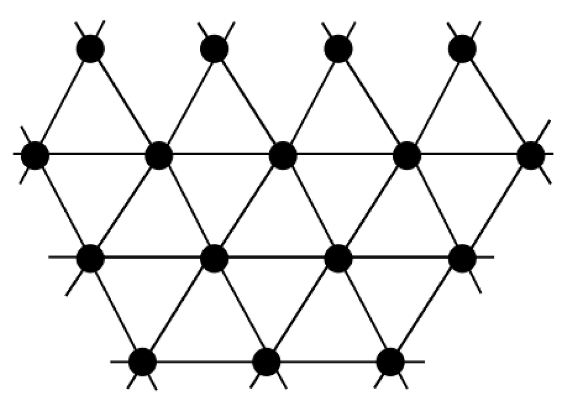

Fig. 1: An Infinite Graph

Central Question. Assume that (G, w_t) is a.s. recurrent (resp. transient) for each $t \geq 0$. When is $\{\langle X_t, w_t \rangle\}_{t=0}^{\infty}$ also recurrent (resp. transient)?

Definition 2 (Recurrence/Transience)**.** An RWCE on $G = (V, E)$ is *recurrent* if every vertex is visited infinitely often almost surely. It is *transient* if every vertex is visited finitely often almost surely. Otherwise, the RWCE is of *mixed-type*.

The following example shows that nonelliptic RWCEs can be neither recurrent nor transient.

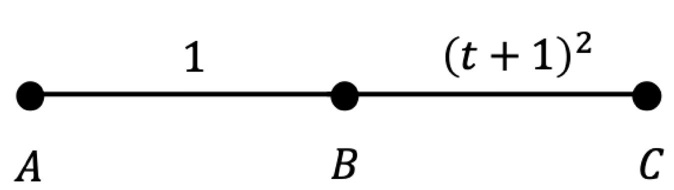

Fig. 2: A Nonelliptic RWCE

Let $X_0 = A$ and $E_t = \{X_{2t+2} = A\}$. Since

$$\sum_{t=0}^{\infty} \frac{1}{1 + (t+1)^2} < \infty,$$

by Borel-Cantelli, the RWCE eventually never visits vertex A.

Definition 3 (Elliptic RWCE)**.** Let $\{\langle X_t, w_t \rangle\}_{t=0}^{\infty}$ be an RWCE on $G = (V, E)$. We say it is *elliptic* (uniformly in time) if $\mathbb{P}(X_{t+1} = y \mid X_t = x)$, whenever well-defined, is bounded away from 0 as t varies.

A special case is when $\{w_t\}_{t=0}^{\infty}$ is *monotone* and *bounded*.

Theorem 1 (Amir et al.)**.** *Let $\{\langle X_t, w_t \rangle\}_{t=0}^{\infty}$ be any increasing (resp. decreasing) RWCE on a tree T bounded above (resp. below) by w_∞ such that (G, w_∞) is recurrent (resp. transient). Then, $\{\langle X_t, w_t \rangle\}_{t=0}^{\infty}$ is also recurrent (resp. transient).*

Conjecture 1 (Amir et al.)**.** *For general graphs, there is an analogous version of Theorem 1 if the RWCE is nonadaptive, meaning the distribution of w_{t+1} given $w_0, \ldots, w_t$ is independent of $X_0, \ldots, X_t$.*

Main Result

We take a different perspective and give a result that holds for any elliptic RWCE on any graph that is only "slightly" changing from some deterministic weighted graph. Our result is conveniently stated in terms of resistances, which are simply the reciprocal of weights: Write $r_t := 1/w_t$ for any $t \geq 0$.

Theorem 2. *Let $G = (V, E)$ be any graph and $w_0 : E \to (0, \infty)$ be any deterministic weight function such that (G, w_0) is recurrent (resp. transient). Let $\{\langle X_t, w_t \rangle\}_{t=0}^{\infty}$ be any elliptic RWCE on G such that $\sum_{t,e} |r_t(e) - r_{t+1}(e)|$ is bounded. Then, $\{\langle X_t, w_t \rangle\}_{t=0}^{\infty}$ is also recurrent (resp. transient).*

Proof Overview

Let $\{\langle X_t, w_t \rangle\}_{t=0}^{\infty}$ be the given RWCE on $G = (V, E)$.

1. Attach a single vertex s' to s and construct a new RWCE $\{\langle X'_t, w'_t \rangle\}$ on the new graph.

2. Show that the recurrence of $\{\langle X'_t, w'_t \rangle\}$ implies the recurrence of $\{\langle X_t, w_t \rangle\}_{t=0}^{\infty}$.

⇒The problem reduces to RWCEs on G where s has a single neighbor.

3. Construct a supermartingale on V that involves ratios of voltages in electrical networks with s as the source.

4. Use step 3 and the optional stopping theorem to derive a condition for any elliptic RWCE to be recurrent.

5. Derive an explicit bound on the ratio of voltages.

6. Use step 4 and 5 to show that $\langle X'_t, w'_t \rangle_{t=0}^{\infty}$ is recurrent.

Graphs and Electrical Networks

Assume that some $s \in V$ is fixed as the origin.

Definition 4 (Voltages)**.** Let $V_n := \{v \in V : d(s, v) \leq n\}$ and $\partial V_n := \{v \in V : d(s, v) = n\}$. For $n \geq 1, t \geq 0$, and $x \in V_n$, let $v_{n,t}(x)$ denote the (random) voltage of x in (G, w_t) when s is grounded and ∂V_n is kept at voltage 1.

The **key connection** between random walks on graphs and electrical networks is that $v_{n,t}(x)$ equals the probability that a random walk on (G, w_t) beginning at x will hit ∂V_n before s. In particular, both quantities are *harmonic*, meaning that $v_{n,t}(x)$ is a weighted average of $v_{n,t+1}(y)$ where $y \sim x$.

Definition 5 (Effective Resistance)**.** The effective resistance of (G_n, w) with s given equals

$$R_{\text{eff},n} := \sum_{e \in E_n} i(e)^2 r(e)$$

where i is the amount of flow through e.

Then, a random walk on (G, w) is recurrent if and only if

$$\lim_{n \to \infty} R_{\text{eff},n} = \infty.$$

Main Lemmas

Steps 1 and 2. We reduce the problem to the special case where s has a single neighbor.

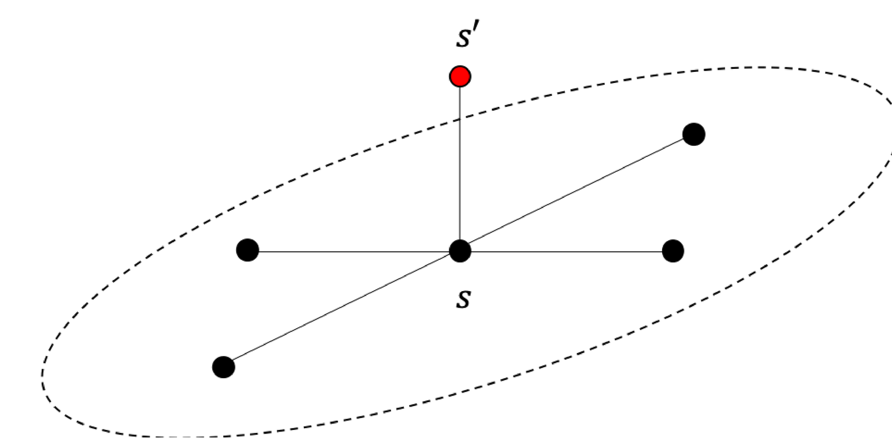

Fig. 3: Attaching s' to s

Steps 3 and 4. We study recurrence by constructing a supermartingale and using the optional stopping theorem.

Lemma 1. *Let $\{\langle X_t, w_t \rangle\}_{t=0}^{\infty}$ be an elliptic RWCE on $G = (V, E)$ where $X_0 = s$. For $n \geq 1$ and $t \geq 0$, let*

$$\alpha_{n,t} := \max_{u \in V_n \setminus \{s\}} \frac{v_{n,t+1}(u)}{v_{n,t}(u)} \geq 1.$$

For each $n \geq 1$, assume there exists $a_n \in \mathbb{R}$ such that a.s. $\prod_{t=0}^{\infty} \alpha_{n,t} \leq a_n < \infty$. If $\limsup_{n \to \infty} a_n < \infty$ and $v_{n,t}(x) \to 0$ almost surely as $n \to \infty$ for any $t \geq 0$ and $x \in V$, then $\{\langle X_t, w_t \rangle\}_{t=0}^{\infty}$ is recurrent.

Step 5. We bound the voltage ratio using the fact that s has a single neighbor.

Lemma 2. *Assume s has a single neighbor x and $w_t(s, x) = 1$ for all $t \geq 0$. Then, for any $n \geq 1, t \geq 0,$ and $u \in V_n \setminus \{s\}$, we have*

$$\left| \frac{v_{n,t+1}(u)}{v_{n,t}(u)} - 1 \right| \leq \sum_{e \in E} |r_t(e) - r_{t+1}(e)|.$$

Step 6. Show that the condition in Lemma 1 is satisfied by using Lemma 2 along with the assumption that $\sum_{t,e} |r_t(e) - r_{t+1}(e)|$ is bounded.

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References

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