Generalizing to the Majority Voting Model for Branching Brownian Motion

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Introduction

Brownian Motion is very closely related with PDEs. It can be easily shown, that $\mathbb{E}_x[u_0(B_t)] = \mathbb{E}[u_0(x + B_t)]$ is a solution to the heat equation with initial condition u_0. **By using Branching Brownian Motion and a tree voting structure, we are able to extend this connection to find a probabilistic model for a big class of reaction-diffusion equations.**

Voting models on Branching Brownian Motion

- Consider an initial condition $0 \ge u_0(x) \le 0$ on the real line.
- We start with one particle at a point $x \in \mathbb{R}$. This particle has an associated exponential random variable with parameter $1/\varepsilon^2$ denoting its branching time τ, and it moves according to Brownian motion until the branching time τ, when it dies and gives birth to $n = 2m + 1$ new, identical particles,and the process repeats from then on.
- At some time t,each particle i that is alive at time t, votes 1 with probability $u_0(X_i)$, where X_i denotes its position,and votes 0 otherwise.
- We propagate these votes back to the initial particle by having a parent particle vote 1 iff the majority of its children voted 1. Using Duhamel's Principle, the probability that the initial particle starting at x voted 1 at time t, $\mathbb{P}_x^t(V_0 = 1) = u(t, x)$ satisfies the PDE

$$u_t = u_{xx} - \frac{u}{\varepsilon^2} + \frac{1}{\varepsilon^2} \sum_{j=0}^{m} \binom{2m+1}{j} u^{2m+1-j} (1-u)^j$$

with initial condition $u_0(x)$.

This is a generalization of the Allen-Cahn Equation $u_t = u_{xx} + u(1 - u)(2u - 1)$.

For our work, we define a function $g(p_1, \ldots, p_{2m+1})$ which gives the probability of a parent node voting 1 given that its children vote 1 with probabilities $p_1, \ldots, p_{2m+1}$. The summation above, which is the nonlinearity we worked with, is the special case in which $p_i = x$ for all i.

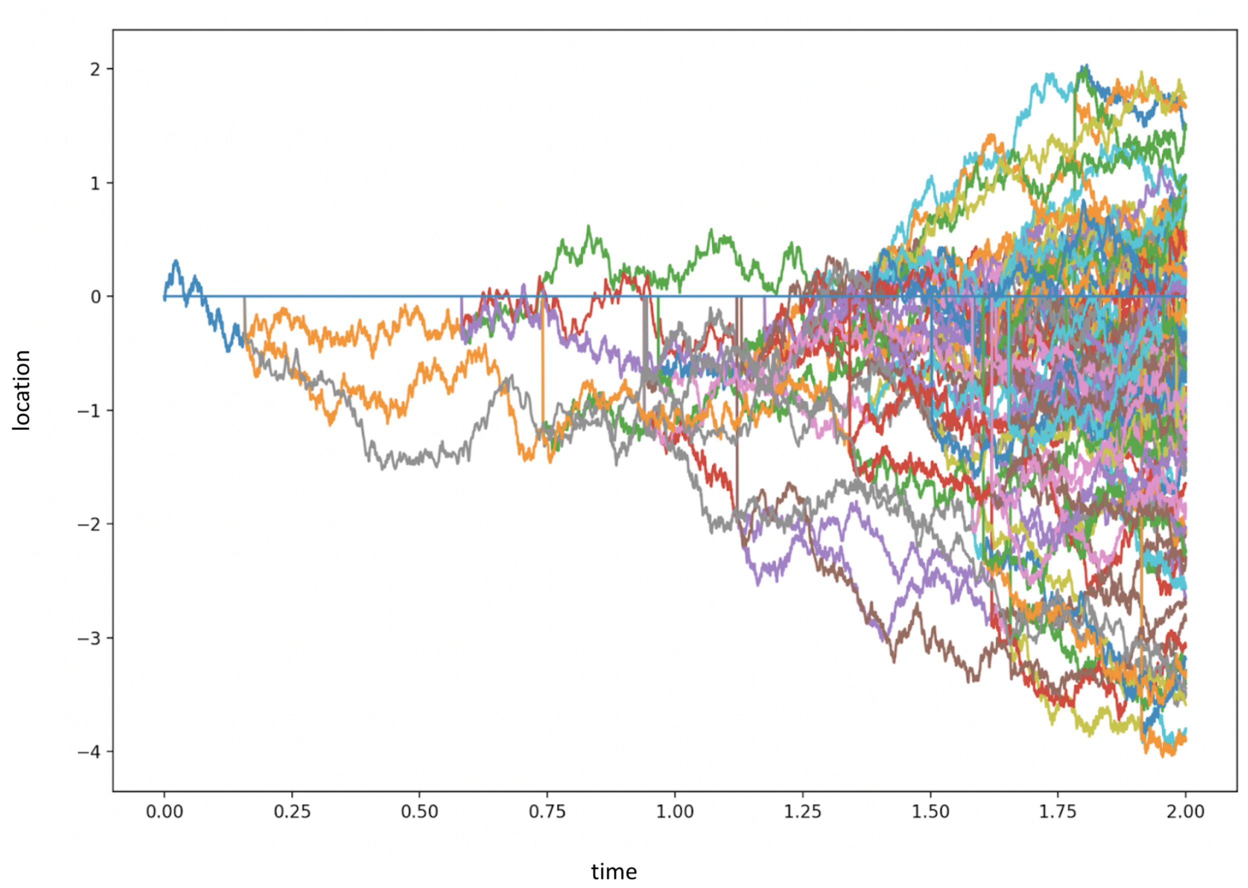

Figure 1: Graphical depiction of BBM with $m = 1$, x-axis is time, y-axis is position.

Main Theorem

The following is a theorem of [1] that we generalize to the case of $2m+1$ children:

Let $T^* \in (0, \infty)$. For all $k \in \mathbb{N}$, there exists $c_1(k)$ and $\varepsilon_1(k) > 0$ such that, for all times $t \in [0, T^*]$ and all $\varepsilon \in (0, \varepsilon_1)$,

1. for $z \ge c_1(k)\,\varepsilon|\log\varepsilon|$, we have $\mathbb{P}_z^t(V_0 = 1) = u(t, z) \ge 1 - \varepsilon^k$
2. for $z \le -c_1(k)\,\varepsilon|\log\varepsilon|$, we have $\mathbb{P}_z^t(V_0 = 1) = u(t, z) \le \varepsilon^k$.

We present both a probabilistic and a PDE proof of this generalized result.

Probabilistic Proof

Lemma 1. For $\frac{1}{2} \le p_1, \ldots, p_{2m+1} \le 1$, $g(p_1, \ldots, p_{2m+1}) \ge \frac{1}{2m+1}(p_1 + \cdots + p_{2m+1})$.

Lemma 2. Given that at time t the tree $\mathcal{T}$ has formed from our branching process, if we started at $z \ge 0$, then the probability of voting 1 is at least the probability of a Brownian motion staying positive; i.e. $\mathbb{P}_z^t(V_0 = 1 \mid \mathcal{T}) \ge \mathbb{P}(z + B_t \ge 0)$.

Lemma 3. ε voting biases at the leaves of our tree become large biases, in $\mathcal{O}(|\log\varepsilon|)$ rounds of voting, so that the initial particle in a regular tree votes 1 with probability $\ge 1 - \varepsilon^k$.

Lemma 4. We define a *regular tree* to be a tree in which every leaf has the same number of nodes in its genealogy. At times t of order at least $\mathcal{O}(\varepsilon^2|\log\varepsilon|)$, large regular trees exist within the tree that formed from our branching process with probability $\ge 1 - \varepsilon^k$

Lemma 5. Brownian particles travel further than $c_1(k)\varepsilon|\log\varepsilon|/2$ in short times with probability $\le \varepsilon^k$.

- We need Lemma 1 to prove Lemma 2.
- From Lemma 5, we are able to conclude that since the particles could not have moved very far in short times, we can use Lemma 2 to say that every leaf has a small voting bias.
- Combining Lemmas 3 and 4, we see that we have a large regular tree with probability $\ge 1 - \varepsilon^k$, and so by Lemma 3, our probability of voting 1 is $\ge 1 - \varepsilon^k$.

PDE Proof

Rewrite our PDE as the equation $u_t = u_{xx} + f(u)$, where the nonlinear term is given by

$$f(u) = \sum_{j=0}^{m} \binom{2m+1}{j} u^{2m+1-j} (1-u)^j - u.$$

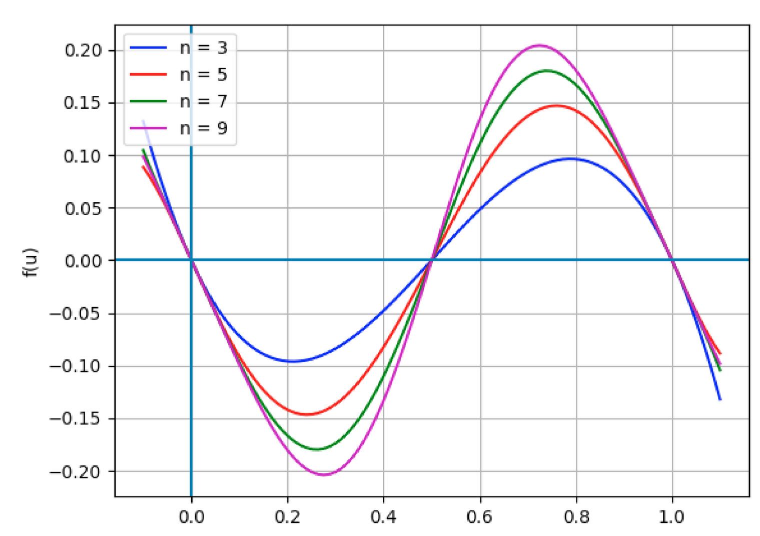

Figure 2: Graphs of $f(u)$ for several values of n.

- In accordance with [2], $f(u)$ is a bistable nonlinearity,so it must admit traveling wave solutions of form $u(t, x) = U(x - ct)$ where U satisfies the second-order ODE:

$$u'' + cu' + f(u) = 0.$$

- U is monotonically increasing, which approaches 1 at ∞ and 0 at $-\infty$.
- The solution $u(t, x)$ converges to a translation of the traveling wave $U(x)$ exponentially fast in time
- The speed of the traveling wave $c = 0$, which means that $u(t, x)$ converges to the steady-state solution.
- We estimate the steady-state solution $U(z)$ and prove $U(z - z_1) \ge 1 - \varepsilon^k$ for $z - z_1 \ge b_1(k)|\log\varepsilon|$ and $U(z - z_1) \le \varepsilon^k$ for $z - z_1 \le -b_1(k)|\log\epsilon|$. We also show that $z_1 = \mathcal{O}(\log\varepsilon)$.
- By a change of variable, we notice that $u(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2})$ satisfies the equation with non-linearity $\frac{1}{\varepsilon^2} f(u)$, which finishes the proof.

Simulations

To get a visual sense for the constants $c_1(k)$ that appear in our work, we numerically simulate the solution to the PDE and determine the $c_1(k)$ thresholds over which our inequalities hold.

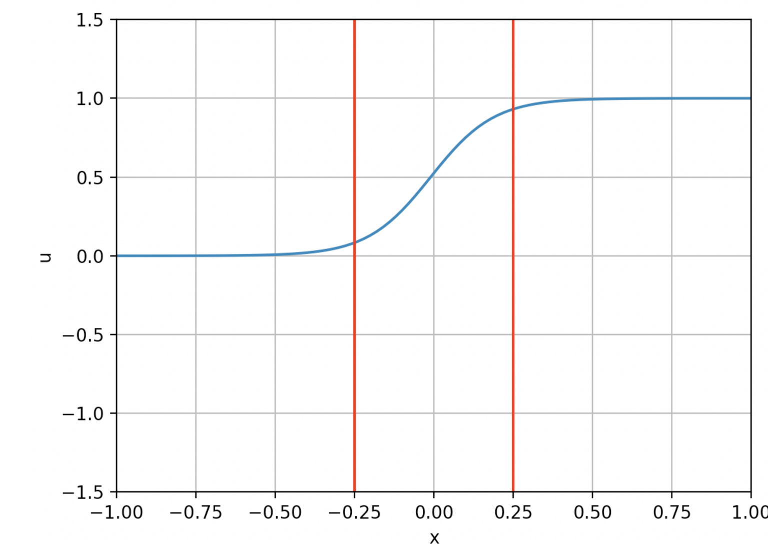

Figure 3: Numeric Allen-Cahn Solution

We accomplish this in two ways. Above, we have shown the simulation achieved by numeric PDE methods; particularly Euler's method for approximating PDEs. Below, we have used the voting model on Branching Brownian Motion to obtain an approximation for our PDE using experimentally obtained probabilities via Monte-Carlo simulations.

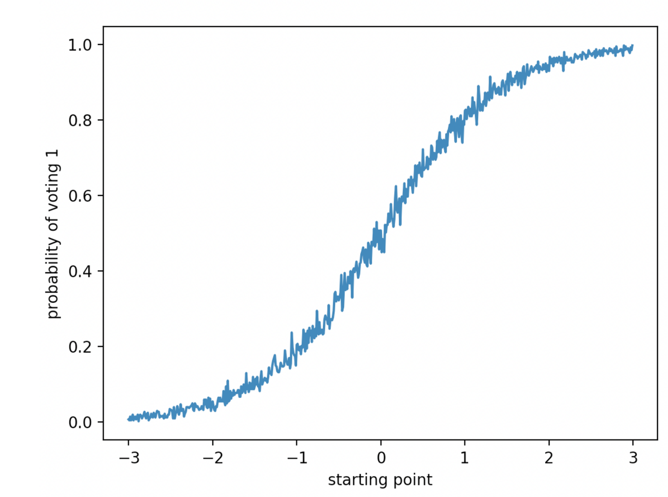

Figure 4: Monte-Carlo Allen-Cahn Solution

Using our simulations for the solutions to Allen-Cahn, we can determine precise values $c_1(k)$ of our main theorem. We will denote by $c_1(k)$ the constant for the $1 - \varepsilon^k$ case and $c_2(k)$ the constant for the ε^k case. For $\varepsilon = 0.1$, we obtain the following table of values:

k	$c_1(k)$	$c_2(k)$
1	1.04230676	1.04230676
2	2.08461351	2.08461351
3	3.12692027	3.12692027
⋮	⋮	⋮

Acknowledgements

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References

[1] Sarah Penington Alison Etheridge Nic Freeman. "Branching Brownian Motion, mean curvature flow and the motion of hybrid zones". In: (2016).

[2] Jimmy Garnier Thomas Giletti Francois Hamel Lionel Roques. "Inside dynamics of pulled and pushed fronts". In: (2011).