On the Weierstrass Elliptic Function as an Exponential Map

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Introduction

Motivating Fact: Elliptic Curves over $\mathbb{C} \Longleftrightarrow$ Complex Tori

1. Elliptic Curves over $\mathbb{C}$:

- Solutions in $\mathbb{C}^2$ to the *Weierstrass Form*

$$y^2 = x^3 + Ax + B$$

for some $A, B \in \mathbb{C}$, plus a *point of infinity* ∞.

2. Complex Tori:

- A lattice $L \subset \mathbb{C}$ gives the complex torus $\mathbb{C}/L$.

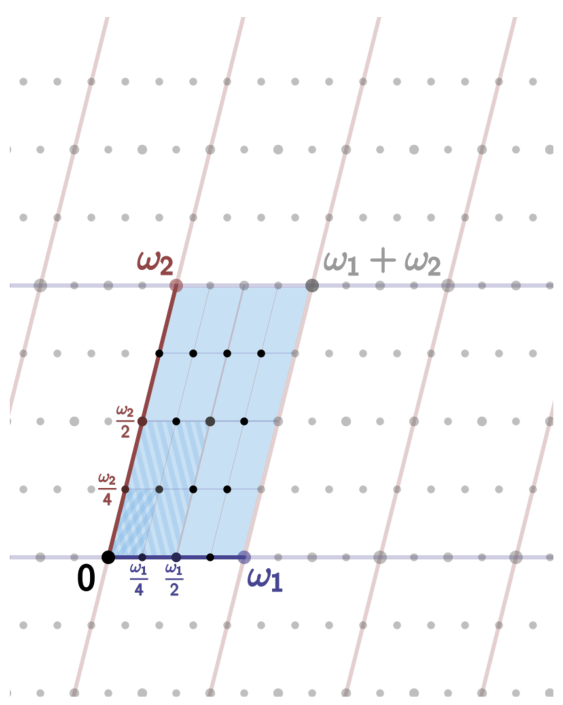


Fig. 1: Complex Torus

Group Law

Any elliptic curve forms a group with identity ∞.

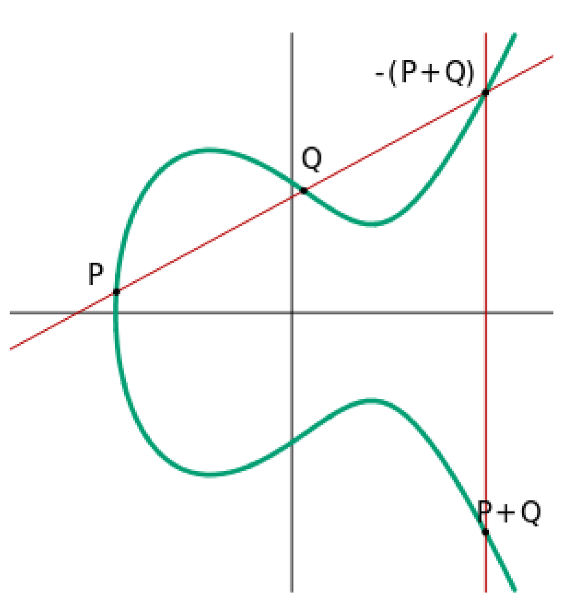


Fig. 2: Group Law

- Group Law: "Intersect and reflect."

Weierstrass Elliptic Function

Let $L \subset \mathbb{C}$ be a lattice. Then,

$$\wp(z) = \frac{1}{z^2} + \sum_{\substack{\omega \in L \\ \omega \neq 0}} \left[\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right]$$

is the Weierstrass elliptic function for L.

Main Objective

Let E be an elliptic curve over $\mathbb{C}$. Then, $E \cong \mathbb{C}/L$ for some lattice L. The standard method to show this uses the Weierstrass elliptic function. That is,

$$z \mapsto (\wp(z), \tfrac{1}{2}\wp'(z))$$

gives a surjective homomorphism between $\mathbb{C}$ and E with kernel L.

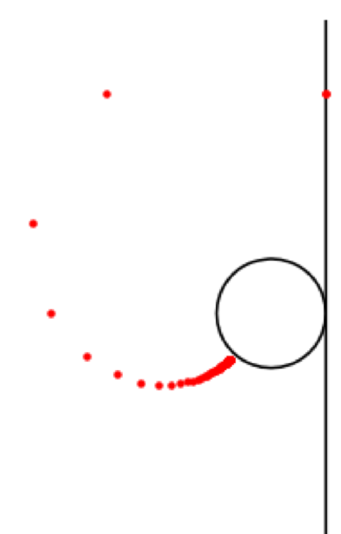
Objective:

- **Find a natural, geometric isomorphism between E and $\mathbb{C}/L$.**
- **Show that $\wp$ is essentially the exponential map on E.**

Toy Problem (Circle)

Q. How can we show $S^1 \cong \mathbb{R}/2\pi\mathbb{Z}$?

A. Construct a surjective homomorphism from a line to a circle!



$$\exp(x) := \lim_{n\to\infty} \left(1 + \frac{ix}{n}\right)^n$$

Fig. 3: Exponential Map

Defining The Exponential Map on Elliptic Curves

Let E be an elliptic curve over $\mathbb{C}$.

1. Consider E in $\mathbb{C}^2$ and pick a point $P \in E$
2. Make P the identity by the new group law

$$P_1 +_{E'} P_2 = P_1 +_E P_2 -_E P$$

3. Take the tangent space to P and associate it with $\mathbb{C}$
4. Define the exponential map

$$\exp(z) := [2^n] f\left(\frac{z}{2^n}\right)$$

where f projects $z/2^n \in \mathbb{C}$ to the elliptic curve, and "$[2^n]f(\frac{z}{2^n})$" means that $f(\frac{z}{2^n})$ is added to itself 2^n times with respect to $+_{E'}$.

Claim: This map is a well-defined surjective homomorphism whose kernel is a lattice L, so it induces an isomorphism between E and $\mathbb{C}/L$, and in fact $\exp(z) = (\wp(z), \frac{1}{2}\wp'(z))$.

Numerical Work

Exponential map for $y^2 = x^3 - 1$ with identity point $(1, 0)$. RGB values indicate argument, opacity indicates norm (zeros in white). The graphs match the behavior of $\wp(z)$ and $\frac{1}{2}\wp'(z)$, respectively, as expected.

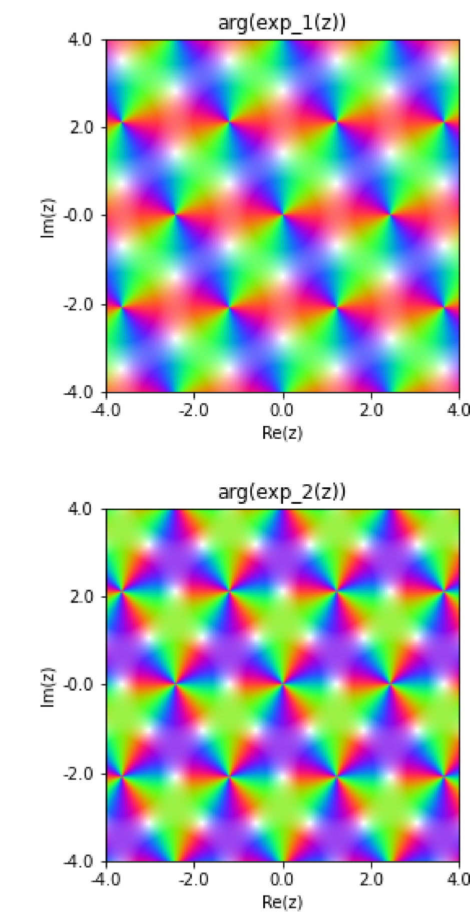


Fig. 4: Plots

Outline of Proof of Claim

- Rewrite $2^n f(\frac{z}{2^n})$ as a telescoping sum.
- Derive estimates showing that $+_{E'}$ is "approximately the same as Euclidian addition" near P.
- Use these estimates to show that the sequence defining $\exp$ converges (i.e. $\exp$ is well-defined) and that $\exp$ is a homomorphism.
- Show that the image of the exponential map is an open and closed subset of a connected set, so $\exp$ is surjective.
- Because E (endowed with an appropriate topology) is compact, conclude by arguing the kernel of $\exp$ must be a lattice L in order for $\mathbb{C}/\ker\exp$ to be compact.
- Compare poles and zeros, take a quotient, and use Liouville's Theorem to show that the exponential map is equal to $z \mapsto (\wp(z), \frac{1}{2}\wp'(z))$.

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